A Formal Analysis of Search Auctions Including Predictions on Click Fraud and Bidding Tactics

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ABSTRACT
This paper presents a model of Paid Search Auctions with very few assumptions. It provides analytical predictions on auction dynamics, bidder optima and equilibria. The model makes a series of testable predictions which may be falsified through future observation.

Categories and Subject Descriptors
J.1 [**Administrative Data Processing**]: *Marketing*

General Terms
Algorithms, Economics, Experimentation, Theory.

Keywords
Pay for placement, Pay per click, Auction, Search Engine, Equilibrium, Optimization

1. INTRODUCTION
The rise of the Search Engine Auction has been remarkable. Paid Search Auctions consumed 40% of on-line advertising in 2004 (Morrissey, 2003). However despite the commercial success of this medium, there has been little formal analysis of the auction design. This article attempts a small foray into this problem. We introduce a formal mathematical model of PPC auctions that uses very few assumptions. Consequently, we are able to make some rather broad (and in some cases surprising) predictions about bidding behavior, equilibrium behavior, and other topics of academic and commercial interest. In addition, we try to cast light on some "hot" topics including Click Fraud, Price escalation, and Bidding wars.

The purpose of this paper is to present the model and predictions in concise form, so that researchers can review these results and perhaps test some of the predictions. A lengthy discourse on the implications of these results, although important, will not be attempted here.

2. AUCTION MODEL
We introduce the following equations:

$$R_k = r_k c_k \quad (1) \qquad p_k = p_k(b_k) \quad (5)$$

$$A_k = a_k c_k \quad (2) \qquad Cost_k = b_k c_k \quad (6)$$

$$\pi_k = (r_k - b_k)\, c_k \quad (3) \qquad CPA_k = Cost_k / A_k \quad (7)$$

$$c_k = c_k(p_k(b_k)) \quad (4) \qquad ROAS_k = R_k / Cost_k \quad (8)$$

where

b_k is the bid price of keyword k.

$c_k(p_k)$ are the clicks generated from keyword k when at position p per unit time. This is a function mapping keyword and position to clicks with the further property that the function decreases monotonically; that is the clicks in position p-1 should be greater than or equal to the clicks in position p, $c_k(p-1) \geq c_k(p)$ (Assumption A1). Sometimes we will shorten this variable to c_k for the clicks generated by a keyword per unit time.

R_k is the revenue generated by a keyword per unit time.

r_k is the expected revenue generated after a customer arrives at your site from a clickthrough on keyword k.

A_k is the number of acquisitions generated by a keyword per unit time.

a_k is the probability of conversion occurring after a customer arrives at your site from a clickthrough on keyword k.

$p_k(b_k)$ is the position that would be awarded if a bid of b_k were placed by the bidder. Sometimes we will shorten this variable to p_k for the position occupied by the bidder on keyword k.

$Cost_k$ incurred by the bidder per unit time on keyword k, which is equal to the number of clicks generated multiplied by the price paid for each click.

π_k is the profit generated on keyword auction k per unit time.

We will use the superscript $^{(p)}$ following a variable to denote a particular position p on the auction. For instance $b_k^{(p)}$ is the price of a position p on keyword auction k. b_k will refer to the bid price that a bidder is paying for keyword k. $^+$ will denote a changed variable. * will denote a variable at an optimum. O is the minimum bid allowed on the auction and o is the minimum amount a bid can be adjusted.

Although we have introduced a very abstract model, we will need to make some assumptions in order to characterize the dynamics of these auctions more faithfully.

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2.1 Assumptions

Assumption A1: Clicks increases with position. Figure 1 shows a typical click versus position curve. Although we believe that clicks is an exponential function of position, we can weaken this relationship a little and merely assume that the clicks in position p will be greater than position p+1. This phenomenon is also likely to occur because of the "list effect" in psychology - a known phenomenon in which subjects read the top of the list first. Thus we assume $c_k(p_k)>c_k(p_k+1)$

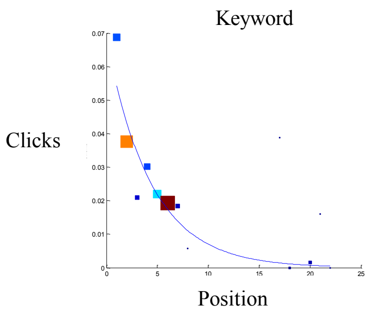


Figure 1: Typical clicks-position curve for one keyword.

Assumption A2: Acquisition rate is independent from position. Some in the search engine marketing community believe that acquisition rates in lower positions are higher than acquisition rates in higher positions. The rationale goes as follows. A customer who clicks through on a listing which is buried in listing 10 or 20 is really looking hard for a product or service. Thus they might show a greater propensity to convert.

As yet we have not yet observed this phenomenon. Figure 2 shows a simple plot of acquisition probability against position. The flat acquisition probability indicates that there is either no or very little relationship. A chi-square test for independence between position and expected acquisition probability also returns a result of no effect ($p < 0.9998$). Hence we assume $a_k \perp p_k$

Assumption A3: Stationarity. It is assumed that the various functions are stationary and not complex functions of time or other variables. Within a discrete period of time, these functions will be close enough to stationary to allow these theorems to hold.

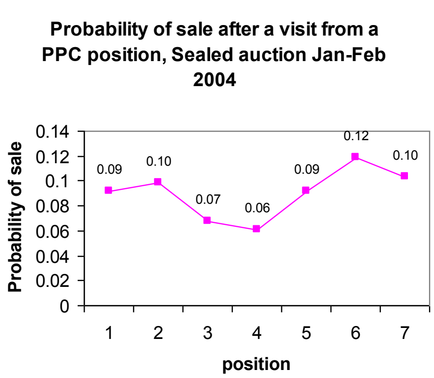


Figure 2: Probability of acquisition versus position.

3. AUCTION DYNAMICS

Before proceeding to prove some of the more interesting results, we present a qualitative picture of how these auctions behave. We have plotted the change in variables with position on auction. These figures are curve-fits to actual bidding data, with the following parametric assumptions:

$$c_k(p_k) = \alpha_k e^{\beta_k p_k} \tag{9}$$

$$p_k(b_k) = \theta_k e^{\varphi_k b_k} \tag{10}$$

Figure 3 shows that ROAS decreases as position approaches 1. Profit, on the other hand, peaks at an optimum point. The steep decline in profit after the optimum is because of our use of an exponential with exponential power to model clicks and position; and we believe it accurately models the phenomenon that if bidders over-shoot their optimum, they experience a rapid decline in profitability.

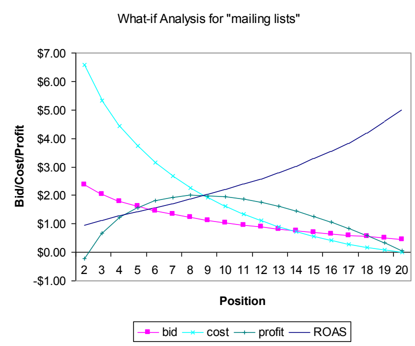


Figure 3: Typical auction

Theorem 1: The Volume-Efficiency Trade-off. Volume as measured by Conversions (Revenue) stays the same or increases if bid increases. Yet efficiency, as measured by CPA (ROAS) remains the same or gets worse if bid price increases.

If you have a keyword k with a bid price of b_k and a current CPA$_k$, and if you increase the bid price to $b_k^+ > b_k$, you will have $A_k^+ \geq A_k$ and CPA$_k^+ \geq$ CPA$_k$

Proof

We can simplify CPA$_k$ as follows:

$$CPA_k = \frac{Cost_k}{A_k} = \frac{b_k c_k}{a_k c_k} = \frac{b_k}{a_k} \tag{11}$$

As b_k increases, p_k either stays the same or moves towards 1. As p_k moves towards 1, c_k increases or stays the same (Assumption A1). Finally, as b_k increases, A_k either increases or stays the same because of (2). For the next part of the proof, consider that as b_k increases CPA_k ostensibly increases as shown in (11), however because of the second price nature of the auction (the winner pays $b_k^{(p-1)}+o$), CPA_k actually either increases or stays the same as shown. Thus on second price auctions if $b_k^+>b_k$ then $A_k^+\geq A_k$ and $CPA_k^+\geq CPA_k$

Pay per click auctions are often difficult for new users to grasp because of this phenomenon of declining efficiency. This is like paying $0.05 to get 5 apples, $1 per apple to get 10 apples, and

$100 per apple to get 11. Never-the-less, this scenario isn't completely unheard of in the world of marketing. In surface mail and telemarketing campaigns, it often costs more money per customer to obtain a greater reach in the population, as is evidenced by newspapers with greater national coverage commanding higher advertising space costs per issue than regional newspapers.

4. OPTIMA

The objective of bidders can vary, but in our experience bidders generally aim for one of the following goals:

1. Maximize Impressions constrained by budget
2. Maximize Clicks constrained by budget
3. Maximize Revenue constrained by ROAS and budget
4. Maximize Conversions constrained by CPA and budget
5. Maximize Profit constrained by budget

Clicks and impressions objectives are very rare today. All clients that iProspect manages are using well-defined revenue, conversions and profit objectives as shown in figure 4.

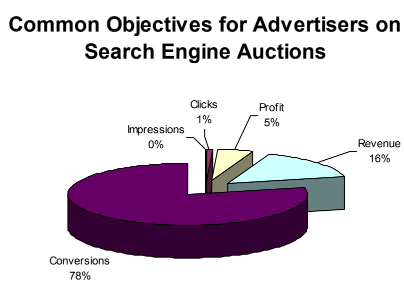


Figure 4: Objectives used by the set of clients managed by iProspect.

Proposition 1: Equivalence of Revenue maximization subject to ROAS and Conversions maximization subject to CPA. Formulaically these objectives and constraints are identical except that $r_k = a_k$ and $ROAS_k=1/CPA_k$. Thus the following theorems that apply to conversions maximization subject to CPA also apply to revenue maximization subject to 1/ROAS.

Theorem 2: Bid that Maximizes Conversions subject to CPA for a Single Listing. Let CPA_k^* be the Cost Per Acquisition that the user desires for keyword k. The price b_k^* that maximizes conversions subject to CPA_k^* is $a_k\, CPA_k^*$

Proof

Let CPA_k^* be the Cost Per Acquisition that the user desires for keyword k. According to the Volume-Efficiency Theorem, Conversions increase (or stay the same) as CPA increases. Therefore, the point at which the maximum number of conversions will be generated, subject to the CPA being below CPA_k^*, is the point at which the actual CPA is equal to the CPA_k^*. Thus, all that remains to maximize conversions subject to CPA, is to find the bid which will allow the CPA_k^* to be achieved. Since $CPA_k^* = b_k^* / a_k$, this can be calculated as follows

$$b_k^* = a_k CPA_k^* \quad (12)$$

Corollary 2: Bid that maximizes revenue subject to ROAS is

$$b_k^* = \frac{r_k}{ROAS_k^*} \quad (13)$$

Theorem 3: Bid that Maximizes Unconstrained Profit for a Single Listing. The keyword optimum for profit depends upon the shape of the c_k and p_k functions. Consequently we need to know those functions to solve for b_k^*. In lieu of that, we can state that the maximum lies between $b_k^* \in [0, r_k]$, since these are zero points for profit.

Theorem 4: Bid that minimizes CPA (maximizes ROAS) for a Single Listing. A by-product of equation 11 is that the minimum CPA (maximum ROAS) will be generated when the bid price is the smallest, or $b_k^*=O$.

4.1 Global Optimum

Theorem 5: Global Optimum is Same as Local Optimum when Objectives are Unconstrained. For unconstrained profit, conversions or revenue, the global optimum bid prices are given by setting each keyword to its locally optimum bid price.

Proof

$$\pi = \sum_{k=1}^{K} \pi_k \;;\; \pi_k^* \geq \pi_k^{(p)} \forall p \;\therefore\; \pi^* = \sum_{k=1}^{K} \pi_k^* \quad (14)$$

Theorem 6: Global Optimum may be different from Local when Objective is Conversions subject to CPA. The global optimum for Conversions subject to CPA b_k^* may not be the same as the keyword optimum for Conversions subject to CPA b_k^+; $b_k^* \neq b_k^+$

Proof

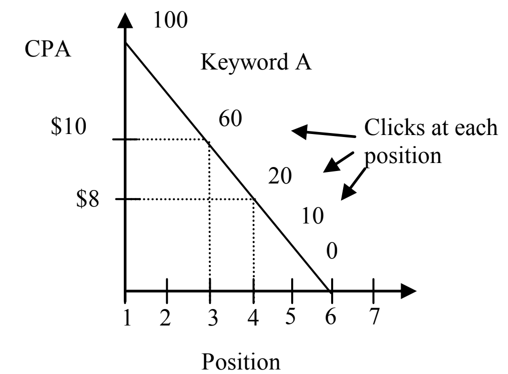
The proof is by example. We imagine that we have a keyword (Keyword B) in which a crowd of competitors are one penny apart from each other from $5.10, $5.09, …, $5.04 for positions 1 to 7 respectively. This scenario is certainly not unusual – indeed it is very common. The conversion rate for keyword B is 50%, and so the CPA for the top 6 positions ranges from $10 to $10.20.

Now consider that we have another keyword (Keyword A). The bidding prices for position 3 and 4 on keyword A are $5 and $4 with a 50% conversion rate.

If we calculate the optimum conversions subject to CPA for each keyword, we would set each keyword to its maximum CPA, which would mean position 6 (CPA $10) on keyword B, and position 3 (CPA $10) on keyword A. The conversions generated will equal 34.

An optimal solution would observe that a large number of conversions exist just above the $10 CPA threshold on keyword B. ie. because keyword B has positions that are just a few pennies away, it is actually cost effective to spend the few pennies to capture those high positions, and soak up all of those conversions.

Thus, the optimal solution under-bids on keyword A to make funds available on keyword B in order to reach the top of that auction. The result is that on keyword B we can pick up 100 conversions at \$10.20 per conversion, and keyword A 10 conversions at \$8 per conversion, for a total CPA of \$10. This is 3 times the number of conversions that could be generated under the local keyword optimum solution (110 versus 32). This important property of the global conversions (revenue) optimum subject to CPA (ROAS) has also been noted by [2].



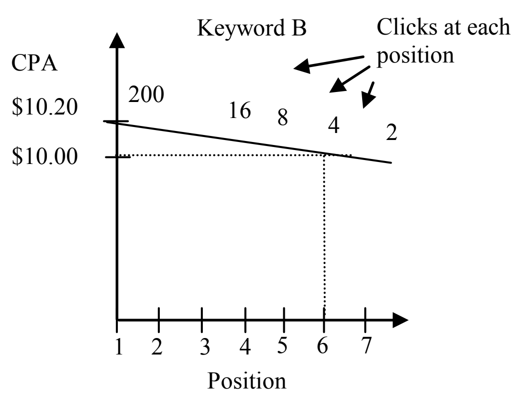


Figure 5: Example showing the difference between local and global optimization.

Assertion: We state without proof that imposition of Budget and ROAS constraints may also result in non-separability.

5. ANALYSIS OF CURRENT TRENDS

5.1 Click Fraud

Theorem 7: Elimination of Click Fraud by Search Engines (Ryan's Theorem): Assuming (a) all bidders are maximizing conversions (revenue) subject to their CPA (ROAS) constraint, (b) assuming click fraud affects all bidders equally, *the effect of the elimination of all Click Fraud by the search engines will neither increase conversions nor decrease CPA for individual bidders* (Meech, 2004).

Proof

Let "click fraud" refer to the practice of clicks being generated on a site without conversions. Let the superscript ^{F} denote variables that are observed under conditions of fraud. We now have some clicks which we would receive were we to remove the fraud c_k, and others that are fraudulent (c_k^F-c_k).

$$A_k^F = a_k^F c_k^F = a_k c_k + 0(c_k^F - c_k) = a_k c_k = A_k \qquad (15)$$

The acquisitions generated under fraudulent conditions are equal to the same generated under normal conditions. If all bidders are maximizing conversions subject to a desired CPA_k^* then their bid price will equal

$$b_k^F = a_k^F CPA_k^* \qquad (16)$$

If we now insert this new bid price into the CPA equation, we have:

$$CPA_k^F = \frac{b_k^F c_k^F}{a_k^F c_k^F} = \frac{\left(a_k^F CPA_k^*\right) c_k^F}{a_k^F c_k^F} = CPA_k^* \qquad (17)$$

Thus, assuming all bidders are affected equally by fraud, and all bidders use single-listing conversions / CPA optimization, the CPA and Conversions generated under fraudulent conditions will be identical to the CPA and Conversions generated under legitimate conditions. The only change will be the click price, acquisition rate (higher) and volume of clicks recorded (lower).

5.1.1 Example

Assume 1 click in 10 generates a conversion, 1 in 10 clicks is fraudulent, and desired CPA is \$10. Under fraudulent conditions we have a_k^F=0.1, CPA_k^*=\$10, c_k^F=10. Because the bidder is adhering to their CPA, they set their bid prices as follows: $b_k^F = a_k^F\ CPA_k^*$= \$1.00. $A_k^F = a_k^F c_k^F$ = 1. If we now eliminate the fraudulent click, we have a_k=0.1111, CPA_k^*=\$10, c_k=9. The bidder now re-adjusts their bid price to take advantage of the higher conversion rate, and their price becomes $b_k = a_k\ CPA_k^*$= \$1.11. Yet with the fraud removed, we still have A_k=1 and CPA_k=\$10.

5.1.2 Discussion

The Click Fraud result is rather startling, and suggests that calls for Google to be more aggressive in pursuit of click fraud may be a little premature. Despite this, click fraud may still have an impact on bidder performance for two reasons.

First, the presence of click fraud will reduce the bid price that a bidder can afford. Because search engines such as Google have minimum bids *O*, if click fraud were to drive prices below this important threshold, it would drive bidders to prices that are under the minimum bid, and so the bidders would drop off the auction. Consequently, the Search Engines (much more so than the bidders!) have an incentive to eliminate click fraud for two reasons: (a) it defrauds them of revenue, and (b) it drives advertisers off keyword auctions, after a sufficient level of contamination.

Second, Ryan's Theorem assumes that click fraud affects competitors equally. If click fraud does not affect competitors equally, then a competitor will develop an advantage if they address and eliminate their click fraud problem while other bidders do not.

5.2 Advertiser Tactics

Bidding is not the only tactic available to advertisers that want to generate more conversions. Advertisers can improve their performance through advertising creative, negative matches, landing pages, keyword discovery, site conversion architecture. The impact of these strategies can vary, and we will examine two general types - site conversion improvement, which will be assumed to improve conversion rates, and negative matching, which will be assumed to prevent non-converting traffic from reaching the site.

Theorem 8: Impact of conversion rate improvement on conversions and CPA. Increase in conversion rate by a factor of G will result in a reduction in CPA by G and increase in conversions by G

Proof

The new number of acquisitions A^+ and Cost Per Acquisition CPA^+ are below:

$$A_k^+ = Ga_k c_k = GA_k \qquad (18)$$

$$CPA_k^+ = \frac{b_k}{Ga_k} = \frac{CPA_k}{G} \qquad (19)$$

Theorem 9: Impact of conversion rate improvement on maximum bid prices. Increase in conversion rate by a factor of G will allow the bid price b_k to be increased by a factor of G whilst still achieving the CPA

Proof

Setting the new price to $b_k^+ = Gb_k$ will let equation (19) equal $CPA_k^+ = CPA_k$.

Theorem 10: Impact of Negative matching. Let $1/G$ is the proportion of traffic that was allowed through to the site because of newly implemented negative matches. Best case impact of negative matching is to reduce CPA by a factor of G, and incur no reduction in conversions.

Proof

Under best-case scenario, all traffic turned away by negative matching is "non-converting". For example, G=3 means that only 1/3rd of the original traffic is now allowed through to the site, and all of the 2/3rd of traffic that was turned away would have reached the site (so incurring the cost) and would then failed to have converted. Under these conditions we have:

$$A_k^+ = Ga_k c_k / G = A_k \qquad (20)$$

$$CPA_k^+ = \frac{b_k c_k^+}{a_k^+ c_k^+} = \frac{b_k c_k / G}{Ga_k c_k / G} = \frac{CPA_k}{G} \qquad (21)$$

5.3 Price

5.3.1 Second Price Bid reveals competitor revenue

Theorem 3 showed that under an ROAS≥1 constraint, Bidders must bid at or below their revenue per click $b_k \leq r_k$ Consequently, assuming all bidders are bidding profitably, the bid price b_k is a lower bound on the amount of revenue each competitor on the auction is generating, per click. It is difficult to verify whether bidders are indeed bidding profitably, but there seems to be evidence that this is the case. Figure 6 shows a collection of prices for the phrase "home loan <state abbreviation>". It shows that states with generally higher home values also have higher click prices. This is consistent with rational bidding.

Keyword	Price	Keyword	Price
Home loan hi	$2.46	Home loan mt	$4.03
Home loan al	$3.23	Home loan tx	$5.26
Home loan co	$3.78	Home loan pa	$6.11
Home loan ky	$3.83	Home loan fl	$6.99
Home loan ks	$3.93	Home loan ca	$7.83
Home loan nh	$3.99	Home loan ny	$9.50

Figure 6: Prices for "home loan <State>" on Overture auction, June 2004. This shows that states with more competitive housing markets command higher click prices.

Even unusually large prices may be explicable when the economics are analyzed. The price for "Mesothelioma" in 2004 peaked at $100 per click (Vise, 2004). Indeed there were 4 companies tied for $100 per click in positions 1 through 4! "Mesothelioma" is a degenerative lung disorder caused by Asbestos exposure. Law firms specializing in settlements locate mesothelioma victims via search engines and bring their cases to trial. The payout from these settlements averages (USD) $1 million with 40% attributable to the law firm. Let's assume 1 person in 10,000 who clicks on the Mesothelioma listing is a case that will win at trial. Plugging this into our equations $r_k = R_k/c_k$ we have: r_k = 400000 / 10000 = $400. In other words, the four competitors crowded at $100 are still below their maximum price of r_k = $400, and so are profitable.

5.3.2 Why are prices increasing?

Many factors may be driving the increase in bid prices. However, a very simple factor, known to affect other auctions, may be the increase in bidders that has occurred over the past several years. In 2004 there were approximately 150,000 advertisers on Google. In 2005 there are approximately 200,000. If we consider that each bidder will have a uniformly distributed maximum profitable bid price given CPA, ROAS or profitability constraints, $b_k^{(p)} \in [0..M]$ where M is the auction ceiling such as $100, then as new bidders arrive on the auction, it is unlikely that position p will escape a price increase. If the auction starts with a price of $b_k^{(p)}=B$, then as more bidders arrive, the probability of the price not going up will equal $(B/M)^P$ - a function which asymptotes toward 0 exponentially fast in P. The hypothesis that prices are being driven by number of advertisers is also supported by some statistics. We examined 24,048 *Yahoo!* keywords in March of 2005, and found the rank correlation between number of advertisers on the auction and position 1 bid price is 0.89 (Figure 6-2). If this is indeed a major factor driving price, then price growth should be roughly indexed to advertiser growth, and if there is a downturn in new advertisers, price growth should also slow.

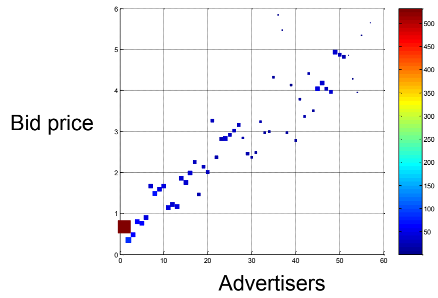


Figure 6-2: Position 1 price of keyword versus number of advertisers for Insurance industry keywords. The size of squares and color represent the number of keywords with approximately this price and number of advertisers.

Metric	R
Pearson Correlation; bid and advertisers	0.6734
Pearson Correlation; ln(bid) and ln(advertisers)	0.8318
Spearman Correlation; bid and advertisers	0.884

Figure 6-3: Keywords with more bidders have higher position 1 prices.

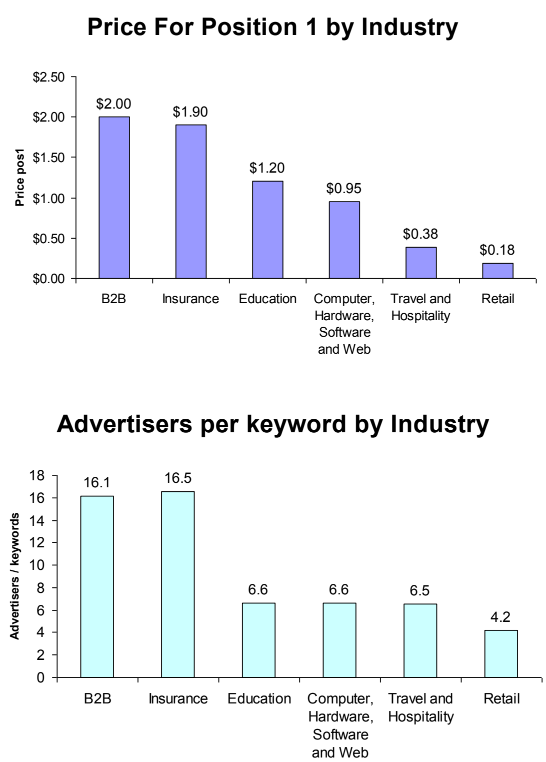


Figure 7: Industries with large numbers of advertisers **(bottom)** also have high position 1 bid prices **(top)**. The most crowded industry is Insurance, where-as the least crowded is Retail.

5.3.3 Why did prices fall after Christmas?

Some private companies have monitored search engine auction prices over time. iProspect has been tracking prices since November 2003, and Fathom Online has been tracking prices publicly since October 2004. Much has been made of monthly price increases, and it seems that every month a news story on click price makes headlines [3]. There was recently a stir because while December 2004 click prices had increased, January 2005 click prices *decreased* [1]. This is also visible in our keyword price data (Figure 8 top). This is rare indeed, given that click prices are increasing at an annual rate of about 35% [2; Figure 8]. What could have caused this decline?

Theorem 8 showed that if conversion rate (revenue rate) could be increased by a factor of *G*, bid price could also be increased by *G* whilst still achieving the same ROAS and CPA as before. Thus we hypothesize that bid prices will track conversion rate. We believe that this is what happened in Christmas 2004.

Conversion rates did indeed increase over the Christmas period before collapsing in January (Figure 8 bottom). We hope that further observations might better illustrate this correlation between conversion rate and bid price.

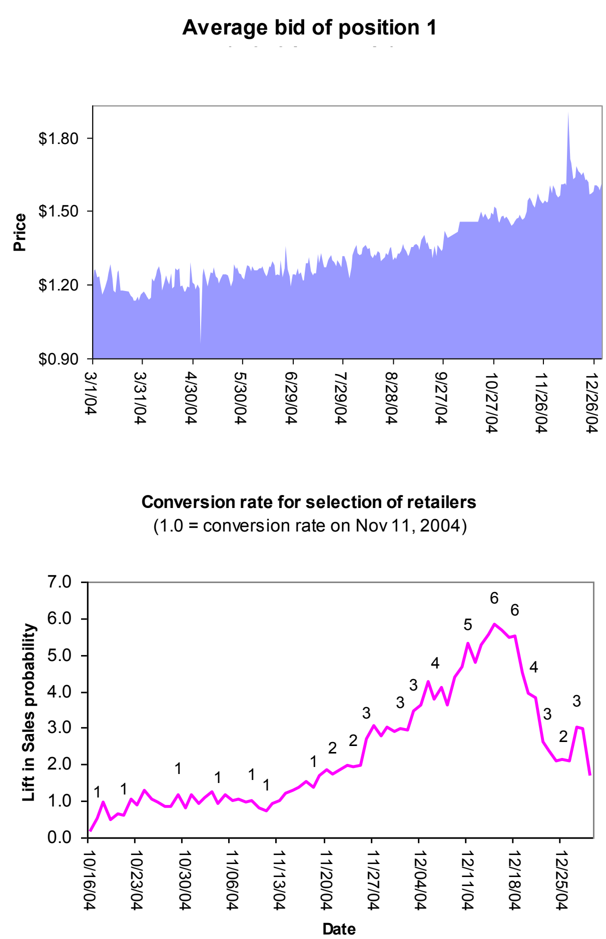


Figure 8 (top): Overture keyword prices increased at Christmas 2004 and then dropped in January 2005. We conjecture that this is a result of an increased conversion rate. **(bottom)** Conversion rate for a selection of retailers over the Christmas period.

5.3.4 The impact of price escalation on bidders

Many advertisers think that increasing keyword prices will lead to keyword auctions becoming unprofitable. This is not true. If an advertiser were profitable on a keyword before a price increase, and the position price increases, leaving them at the same price, that advertiser will remain profitable afterwards. After a price increase, the bidder is pushed down into a lower position. At this new position, their price is the same, but they acquire fewer clicks (Assumption A1) and due to the drop in clicks they also acquire fewer conversions and revenue. Yet the profit equation $(r_k-b_k)c_k$ remains at 0 or above. So too, CPA remains the same and so does ROAS.

6. EQUILIBRIUM

Definition 1: Insertion

Say that a new competitor n arrives on an existing auction and begins bidding at price $b_k^{(n)}$. We will refer to this as an *insertion*. Bidders above $b_k^{(n-1)}$ will be unaffected by the arrival of bidder in position n, and so we say that these bidders are *stable*. Insertion affects the bidder at $b_k^{(n-1)}$ or below the deletion point n. We say that some number of these bidders may now be *unstable* - they may no longer be at their optimum profit position and will try to adjust their bid to achieve optimum profit again. We note the following relationship between the pre-insertion and post-insertion performance of each competitor j, where ' refers to a variable after the insertion.

If $b_k^{(j)}>b_k^{(n-1)}$ then	If $b_k^{(j)}=b_k^{(n-1)}$ then	If $b_k^{(j)}<b_k^{(n)}$ then
$c_k^{(j)'}=c_k^{(j)}$	$c_k^{(j)'}=c_k^{(j)}$	$c_k^{(j)'}=c_k^{(j+1)}$
$b_k^{(j)'}=b_k^{(j)}$	$b_k^{(j)'}=b_k^{(n)}+o$	$b_k^{(j)'}=b_k^{(j)}$
$\pi_k^{(j)'}=\pi_k^{(j)}$	$\pi_k^{(j)'}=(r_k^{(j)}-b_k^{(n)}+o)\,c_k^{(j)}$	$\pi_k^{(j)'}=(r_k^{(j)}-b_k^{(j)})\,c_k^{(j+1)}$

Definition 2: Deletion

Say that a competitor n with bidding price $b_k^{(n)}$ departs from this auction. We will refer to this as a *deletion*. Bidders above $b_k^{(n-1)}$ are unaffected by the departure of bidder in position n, and so we say that these bidders are *stable*. Deletion affects bidders below the deletion point n or the competitor immediately above the insertion point at $b_k^{(n-1)}$. We say that some number of these bidders may be *unstable* - they may no longer be at their optimum profit position. The ' superscript will denote a variable after deletion. We note the following relationship between pre-deletion and post-deletion performance of each competitor i.

If $b_k^{(i)}>b_k^{(n)}$ then	If $b_k^{(i)}=b_k^{(n-1)}$ then	If $b_k^{(i)}<b_k^{(n)}$ then
$c_k^{(i)'}=c_k^{(i)}$	$c_k^{(i)'}=c_k^{(i)}$	$c_k^{(i)'}=c_k^{(i-1)}$
$b_k^{(i)'}=b_k^{(i)}$	$b_k^{(i)'}=b_k^{(n+1)}+o$	$b_k^{(i)'}=b_k^{(i)}$
$\pi_k^{(i)'}=\pi_k^{(i)}$	$\pi_k^{(i)'}=(r_k^{(i)}-b_k^{(n+1)}+o)\,c_k^{(i)}$	$\pi_k^{(i)'}=(r_k^{(i)}-b_k^{(i)})\,c_k^{(i-1)}$

Theorem 11: Upper bound on profit optimum following deletion. After a deletion at $b_k^{(n)}$ the profit maximum for competitor $i : b_k^{(i)}\leq b_k^{(n-1)}$ is bounded from above by $b_k^{(n-1)}$.

Proof

$$\pi_k^{(i)*} \geq \pi_k^{(j)}\ \forall j \in [1..n-1] \text{ (by definition of } \pi_k^{(i)*})$$
$$\text{if } b_k^{(i)}<b_k^{(n)} \text{ then } \pi_k^{(i)'} \geq \pi_k^{(i)} \text{ since } c_k^{(i)'}\geq c_k^{(i)}$$
$$\text{if } b_k^{(i)}=b_k^{(n-1)} \text{ then } \pi_k^{(i)'} > \pi_k^{(i)} \text{ since } b_k^{(i)'}< b_k^{(i)}$$
$$\text{if } b_k^{(i)}>b_k^{(n)} \text{ then } \pi_k^{(i)'} = \pi_k^{(i)}$$
$$\therefore \pi_k^{(i)*'} \geq \pi_k^{(i)*} \geq \pi_k^{(j)}\ \forall j \in [1..n-1]$$

Theorem 13: Speed of return to equilibrium after deletion. After a deletion occurring at $b_k^{(n)}$ it will take at most $0.5P^2$ moves for competitors to resume their equilibrium state, P being the number of competitors on the auction, assuming bidders that bid rationally and maximize profit.

Proof

Each competitor i below the deleted item $b_k^{(i)}<b_k^{(n)}$, and the competitor immediately above $b_k^{(n-1)}$, will re-calculate their profit, and will either continue to be in an optimal position at $b_k^{(i)}$, or will need to move to a new optimal profit bid $b_k^{(N)}\leq b_k^{(n-1)}$ (from Theorem 11). If competitor i moves to position N such that $b_k^{(N)}>b_k^{(i)}$ then competitors $j\in[N..i-1]$ will become stable again, because i has returned these competitors to their previous state of having an additional competitor above them. Thus, if any competitors move to a higher bid price, it results a stabilization for i plus some number of additional stabilizations. If a competitor moves to a lower price, then it only results in a single stabilization - the competitor i itself. Thus, the worst-case behavior will be when each unstable competitor at position p moves down the minimum amount - that is to the next position p+1. The maximum number of such moves will be P moves in the first set of shifts, then P-1 (since one will now be at minimum price), then P-2, and so on. As such the maximum number of steps until all competitors reach stability is P^2 / 2 steps.

Theorem 14: Speed of return to equilibrium after insertion. Assuming bidders select prices that increase their utility at each step, then after at most **max** $r_k^{(p)}*P/2o$ steps, all competitors will move to a price such that profit is maximized, and no further actions can increase profit.

Proof

A bidder with bid $b_k^{(p)}$ can change their price to any value between $b_k^{(p-1)}-o$ and $b_k^{(p+1)}+o$, and have no effect on their own utility. Such an action would however change the price and utility for the bidder at position p-1 above them. Since this type of bid change would have no effect on the bidder's own utility, we assume that bidders would not take such an action. After an insertion at n, the prices for all positions at n+1..P will increase and become equal to one position above. The price for n-1 will also increase, equaling $b_k^{(n)}+o>b_k^{(n+1)}+o$. The profit for positions 1..n-2 will not be affected, and since the profit for positions n-1..P will be lower, none of the bidders in positions 1..n-2 will move into those lower positions. However, some bidders $b_k^{(i)}<b_k^{(n)}$ may be unstable after an insertion. If any such competitor i moves into a lower position j, the bidders from i..j-1 will return to their pre-

insertion state, and the price of the position this bidder moves to will increase by o. If any bidder moves up to a more expensive bid, they will similarly increase the position price by o. Thus, in any case, the price of at least one position must increase by at least o. However, every bidder has a finite bid price $r_k^{(i)}$ that they cannot exceed. Consequently, since position price will monotonically increase after each position change, either all bidders will find their maximum profit positions and no further changes will occur, or eventually position prices will increase above $r_k^{(i)}$ and bidders will no longer be able to increase prices. Thus bidding changes must halt after a finite number of steps. The longest it will take for bidding to halt is if each competitor increases their price by the minimum amount. This will equal **max** $r_k^{(p)} * P/2o$ steps.

Corollary 3: If max $r_k/o > P$ then speed of return to equilibrium after deletion is faster than insertion. This follows from theorem 13 and 14. This is an experimentally testable assertion.

Theorem 15: Nash equilibrium: Let an action profile $b_k^{(1..P)}$ be a vector of prices paid by each competitor. This action profile will form a Nash equilibrium if and only if every player's bid is a best response to the other players' actions. A best response $b_k^{(i)}$ is a bid which generates the highest utility for competitor i given that competitor j will choose their own utility maximizing action. (Nash, 1950).

Proof

Proof is inductive. Consider an account with one bidder. No further actions that the bidder can take will increase profit. Thus this auction is trivially at equilibrium.

Consider an auction with C bidders at equilibrium. If a new bidder arrives, then the auction will return to equilibrium within a finite period of time (Theorem 14).

Consider an auction with C bidders at equilibrium. If a competitor leaves the auction the auction will return to equilibrium within a finite period of time (Theorem 13).

Theorem 16: No Guarantee of Nash Equilibrium if Bids that Do Not Increase Utility are Submitted. If there exists one or more bidders who change their prices from $b_k^{(p)}$ to another value $[b_k^{(p+1)}+o..b_k^{(p-1)}-o]$, then the auction may never achieve equilibrium.

Proof

Consider $r_k^{(i)}=100, r_k^{(j)}=10$. Clicks in position 1 and 2 are 1000 and 1 respectively. The competitors start with bids $(b_k^{(i)}, b_k^{(j)})$ = (0.11,0.10). At t=2 these prices change to (0.11,0.12). After t=3 (0.13,0.12). And so on. Eventually we will reach (1.00,0.99). At this time, a further price escalation from j would result in 0 profit, and so it halts the bidding war. j may now execute a bid that has no effect on their own utility, and decrease its bid to one cent above the bid below - $0.10, then we will have (1.00,0.10). Bidder i may do the same thing, and so decrease its bid to (0.11,0.10). At this point we have created a cycle.

7. CONCLUSION

We have presented a simple model of search auctions, including a number of testable predictions. The model can be extended with some additional assumptions. For instance, we are confident that a parameterized exponential function faithfully models the relationship between position and clicks, and bid and position. In this article we have not yet built these assumptions into our results, and tried to make as much progress as possible with the minimum amount of commitments.

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